

OXFORD CAMBRIDGE AND RSA EXAMINATIONS
Sixth Term Examination Papers
administered on behalf of the Cambridge Colleges

MATHEMATICS II

9470

Friday

1 JULY 2005

Morning

3 hours

Additional materials:

- Answer paper
- Graph paper
- Formulae booklet

Candidates may not use electronic calculators

TIME 3 hours

INSTRUCTIONS TO CANDIDATES

- Write your name, Centre number and candidate number in the spaces on the answer paper/ answer booklet.
- Begin each answer on a new page.

INFORMATION FOR CANDIDATES

- Each question is marked out of 20. There is no restriction of choice.
- You will be assessed on the **six** questions for which you gain the highest marks.
- You are advised to concentrate on no more than **six** questions. Little credit will be given to fragmentary answers.
- You are provided with Mathematical Formulae and Tables.
- **Electronic calculators are not permitted.**

This question paper consists of 7 printed pages and 1 blank page.

Section A: Pure Mathematics

- 1 Find the three values of x for which the derivative of $x^2e^{-x^2}$ is zero.

Given that a and b are distinct positive numbers, find a polynomial $P(x)$ such that the derivative of $P(x)e^{-x^2}$ is zero for $x = 0$, $x = \pm a$ and $x = \pm b$, but for no other values of x .

- 2 For any positive integer N , the function $f(N)$ is defined by

$$f(N) = N \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

where p_1, p_2, \dots, p_k are the only prime numbers that are factors of N .

Thus $f(80) = 80\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)$.

- (a) (i) Evaluate $f(12)$ and $f(180)$.
- (ii) Show that $f(N)$ is an integer for all N .
- (b) Prove, or disprove by means of a counterexample, each of the following:
- (i) $f(m)f(n) = f(mn)$;
- (ii) $f(p)f(q) = f(pq)$ if p and q are distinct prime numbers;
- (iii) $f(p)f(q) = f(pq)$ only if p and q are distinct prime numbers.
- (c) Find a positive integer m and a prime number p such that $f(p^m) = 146410$.

- 3 Give a sketch, for $0 \leq x \leq \pi/2$, of the curve

$$y = (\sin x - x \cos x),$$

and show that $0 \leq y \leq 1$.

Show that:

(i) $\int_0^{\pi/2} y \, dx = 2 - \frac{\pi}{2}$;

(ii) $\int_0^{\pi/2} y^2 \, dx = \frac{\pi^3}{48} - \frac{\pi}{8}$.

Deduce that $\pi^3 + 18\pi < 96$.

- 4 The positive numbers a , b and c satisfy $bc = a^2 + 1$. Prove that

$$\arctan\left(\frac{1}{a+b}\right) + \arctan\left(\frac{1}{a+c}\right) = \arctan\left(\frac{1}{a}\right).$$

The positive numbers p , q , r , s , t , u and v satisfy

$$st = (p+q)^2 + 1, \quad uv = (p+r)^2 + 1, \quad qr = p^2 + 1.$$

Prove that

$$\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) + \arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right) = \arctan\left(\frac{1}{p}\right).$$

Hence show that

$$\arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{82}\right) + \arctan\left(\frac{1}{187}\right) = \arctan\left(\frac{1}{7}\right).$$

[Note that $\arctan x$ is another notation for $\tan^{-1} x$.]

- 5 The angle A of triangle ABC is a right angle and the sides BC , CA and AB are of lengths a , b and c , respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r . Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2 . The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R . Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1),$$

where $q = \frac{b+c}{a}$. Deduce that

$$R \leq \frac{1}{\pi(\pi - 1)}.$$

- 6 (i) Write down the general term in the expansion in powers of x of $(1-x)^{-1}$, $(1-x)^{-2}$ and $(1-x)^{-3}$, where $|x| < 1$.

Evaluate $\sum_{n=1}^{\infty} n2^{-n}$ and $\sum_{n=1}^{\infty} n^2 2^{-n}$.

- (ii) Show that $(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \frac{x^n}{2^{2n}}$, for $|x| < 1$.

Evaluate $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n} 3^n}$ and $\sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^n}$.

7 The position vectors, relative to an origin O , at time t of the particles P and Q are

$$\cos t \mathbf{i} + \sin t \mathbf{j} + 0 \mathbf{k} \quad \text{and} \quad \cos\left(t + \frac{1}{4}\pi\right) \left[\frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{k}\right] + 3 \sin\left(t + \frac{1}{4}\pi\right) \mathbf{j},$$

respectively, where $0 \leq t \leq 2\pi$.

(i) Give a geometrical description of the motion of P and Q .

(ii) Let θ be the angle POQ at time t that satisfies $0 \leq \theta \leq 2\pi$. Show that

$$\cos \theta = \frac{3\sqrt{2}}{8} - \frac{1}{4} \cos\left(2t + \frac{1}{4}\pi\right).$$

(iii) Show that the total time for which $\theta \geq \frac{1}{4}\pi$ is $\frac{3}{2}\pi$.

8 For $x \geq 0$ the curve C is defined by

$$\frac{dy}{dx} = \frac{x^3 y^2}{(1+x^2)^{5/2}}$$

with $y = 1$ when $x = 0$. Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{3/2}} + \frac{1}{3}$$

and hence that for large positive x

$$y \approx 3 - \frac{9}{x}.$$

Draw a sketch of C .

On a separate diagram draw a sketch of the two curves defined for $x \geq 0$ by

$$\frac{dz}{dx} = \frac{x^3 z^3}{2(1+x^2)^{5/2}}$$

with $z = 1$ at $x = 0$ on one curve, and $z = -1$ at $x = 0$ on the other.

Section B: Mechanics

- 9** Two particles, A and B , of masses m and $2m$, respectively, are placed on a line of greatest slope, ℓ , of a rough inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between A and the plane is $\frac{1}{6}\sqrt{3}$ and the coefficient of friction between B and the plane is $\frac{1}{3}\sqrt{3}$. The particles are at rest with B higher up ℓ than A and are connected by a light inextensible string which is taut. A force P is applied to B .

- (i) Show that the least magnitude of P for which the two particles move upwards along ℓ is $\frac{11}{8}\sqrt{3}mg$ and give, in this case, the direction in which P acts.
- (ii) Find the least magnitude of P for which the particles do not slip downwards along ℓ .

- 10** The points A and B are 180 metres apart and lie on horizontal ground. A missile is launched from A at speed of 100 ms^{-1} and at an acute angle of elevation to the line AB of $\arcsin \frac{3}{5}$. A time T seconds later, an anti-missile missile is launched from B , at speed of 200 ms^{-1} and at an acute angle of elevation to the line BA of $\arcsin \frac{4}{5}$. The motion of both missiles takes place in the vertical plane containing A and B , and the missiles collide.

Taking $g = 10 \text{ ms}^{-2}$ and ignoring air resistance, find T .

[Note that $\arcsin \frac{3}{5}$ is another notation for $\sin^{-1} \frac{3}{5}$.]

- 11** A plane is inclined at an angle $\arctan \frac{3}{4}$ to the horizontal and a small, smooth, light pulley P is fixed to the top of the plane. A string, APB , passes over the pulley. A particle of mass m_1 is attached to the string at A and rests on the inclined plane with AP parallel to a line of greatest slope in the plane. A particle of mass m_2 , where $m_2 > m_1$, is attached to the string at B and hangs freely with BP vertical. The coefficient of friction between the particle at A and the plane is $\frac{1}{2}$.

The system is released from rest with the string taut. Show that the acceleration of the particles is $\frac{m_2 - m_1}{m_2 + m_1}g$.

At a time T after release, the string breaks. Given that the particle at A does not reach the pulley at any point in its motion, find an expression in terms of T for the time after release at which the particle at A reaches its maximum height. It is found that, regardless of when the string broke, this time is equal to the time taken by the particle at A to descend from its point of maximum height to the point at which it was released. Find the ratio $m_1 : m_2$.

[Note that $\arctan \frac{3}{4}$ is another notation for $\tan^{-1} \frac{3}{4}$.]

Section C: Probability and Statistics

12 The twins Anna and Bella share a computer and never sign their e-mails. When I email them, only the twin currently online responds. The probability that it is Anna who is online is p and she answers each question I ask her truthfully with probability a , independently of all her other answers, even if a question is repeated. The probability that it is Bella who is online is q , where $q = 1 - p$, and she answers each question truthfully with probability b , independently of all her other answers, even if a question is repeated.

- (i) I send the twins the email: ‘Toss a fair coin and answer the following question. Did the coin come down heads?’. I receive the answer ‘yes’. Show that the probability that the coin did come down heads is $\frac{1}{2}$ if and only if $2(ap + bq) = 1$.
- (ii) I send the twins the email: ‘Toss a fair coin and answer the following question. Did the coin come down heads?’ I receive the answer ‘yes’. I then send the e-mail: ‘Did the coin come down heads?’ and I receive the answer ‘no’. Show that the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.
- (iii) I send the twins the email: ‘Toss a fair coin and answer the following question. Did the coin come down heads?’ I receive the answer ‘yes’. I then send the e-mail: ‘Did the coin come down heads?’ and I receive the answer ‘yes’. Show that, if $2(ap + bq) = 1$, the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

13 The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geq 2$) which contains fewer than two errors is denoted by Y . Show that $P(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1 + \lambda)e^{-\lambda}$ and $q = 1 - p$.

Show also that, if λ is sufficiently small,

- (i) $q \approx \frac{1}{2}\lambda^2$;
- (ii) the largest value of n for which $P(Y = n) \geq 1 - \lambda$ is approximately $2/\lambda$;
- (iii) $P(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$.

14 The probability density function $f(x)$ of the random variable X is given by

$$f(x) = k[\phi(x) + \lambda g(x)],$$

where $\phi(x)$ is the probability density function of a normal variate with mean 0 and variance 1, λ is a positive constant, and $g(x)$ is a probability density function defined by

$$g(x) = \begin{cases} 1/\lambda & \text{for } 0 \leq x \leq \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

Find μ , the mean of X , in terms of λ , and prove that σ , the standard deviation of X , satisfies.

$$\sigma^2 = \frac{\lambda^4 + 4\lambda^3 + 12\lambda + 12}{12(1 + \lambda)^2}.$$

In the case $\lambda = 2$:

- (i) draw a sketch of the curve $y = f(x)$;
- (ii) express the cumulative distribution function of X in terms of $\Phi(x)$, the cumulative distribution function corresponding to $\phi(x)$;
- (iii) evaluate $P(0 < X < \mu + 2\sigma)$, given that $\Phi(\frac{2}{3} + \frac{2}{3}\sqrt{7}) = 0.9921$.

