Section A: Pure Mathematics

1 Given that $y = \ln\big(x + \sqrt{(x^2+1)}\big)$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{(x^2+1)}}$.

Prove by induction that, for $n \geqslant 0$,

$$\big(x^2+1\big)\, y^{(n+2)} + (2n+1)\, xy^{(n+1)} + n^2 y^{(n)} = 0 \ ,$$

where $y^{(n)} = \dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ and $y^{(0)} = y$.

Using this result in the case $x = 0$, or otherwise, show that the Maclaurin series for y begins

$$x - \frac{x^3}{6} + \frac{3x^5}{40}$$

and find the next non-zero term.

2 Show that $\cosh^{-1} x = \ln\left(x + \sqrt{(x^2-1)}\right)$.

Show that the area of the region defined by the inequalities $y^2 \geqslant x^2 - 8$ and $x^2 \geqslant 25y^2 - 16$ is $(72/5)\ln 2$.

3 Consider the equation

$$x^2 - bx + c = 0 \ ,$$

where b and c are real numbers.

(i) Show that the roots of the equation are real and positive if and only if $b > 0$ and $b^2 \geqslant 4c > 0$, and sketch the region of the b-c plane in which these conditions hold.

(ii) Sketch the region of the b-c plane in which the roots of the equation are real and less than 1 in magnitude.

4 In this question, the function $\sin^{-1}$ is defined to have domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ and the function $\tan^{-1}$ is defined to have the real numbers as its domain and range $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(i) Let

$$\mathrm{g}(x) = \frac{2x}{1+x^2}\,, \qquad -\infty < x < \infty\,.$$

Sketch the graph of $\mathrm{g}(x)$ and state the range of g.

(ii) Let

$$\mathrm{f}(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)\,, \qquad -\infty < x < \infty\,.$$

Show that $\mathrm{f}(x) = 2\tan^{-1} x$ for $-1 \leqslant x \leqslant 1$ and $\mathrm{f}(x) = \pi - 2\tan^{-1} x$ for $x \geqslant 1$.

Sketch the graph of $\mathrm{f}(x)$.

5 Show that the equation $x^3 + px + q = 0$ has exactly one real solution if $p \geqslant 0$.

A parabola C is given parametrically by

$$x = at^2, \quad y = 2at \qquad (a > 0)\,.$$

Find an equation which must be satisfied by t at points on C at which the normal passes through the point $(h,\ k)$. Hence show that, if $h \leqslant 2a$, exactly one normal to C will pass through $(h,\ k)$.

Find, in Cartesian form, the equation of the locus of the points from which exactly two normals can be drawn to C. Sketch the locus.

6 The plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

meets the co-ordinate axes at the points A, B and C. The point M has coordinates $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right)$ and O is the origin.

Show that OM meets the plane at the centroid $\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right)$ of triangle ABC. Show also that the perpendiculars to the plane from O and from M meet the plane at the orthocentre and at the circumcentre of triangle ABC respectively.

Hence prove that the centroid of a triangle lies on the line segment joining its orthocentre and circumcentre, and that it divides this line segment in the ratio $2:1$.

[The *orthocentre* of a triangle is the point at which the three altitudes intersect; the *circumcentre* of a triangle is the point equidistant from the three vertices.]

7 Sketch the graph of the function $\ln x - \frac{1}{2}x^2$.

Show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - 1}$$

describes a family of parabolas each of which passes through the points $(1\,,\ 0)$ and $(-1\,,\ 0)$ and has its vertex on the y–axis.

Hence find the equation of the curve that passes through the point $(1\,,\ 1)$ and intersects each of the above parabolas orthogonally. Sketch this curve.

[Two curves intersect *orthogonally* if their tangents at the point of intersection are perpendicular.]

8 (i) Prove that the equations

$$|z - (1+i)|^2 = 2 \qquad (*)$$

and

$$|z - (1-i)|^2 = 2\,|z-1|^2$$

describe the same locus in the complex z–plane. Sketch this locus.

(ii) Prove that the equation

$$\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{4} \qquad (**)$$

describes part of this same locus, and show on your sketch which part.

(iii) The complex number w is related to z by

$$w = \frac{2}{z}\ .$$

Determine the locus produced in the complex w–plane if z satisfies $(*)$. Sketch this locus and indicate the part of this locus that corresponds to $(**)$.

Section B: Mechanics

9 B_1 and B_2 are parallel, thin, horizontal fixed beams. B_1 is a vertical distance $d\sin\alpha$ above B_2, and a horizontal distance $d\cos\alpha$ from B_2, where $0 < \alpha < \pi/2$. A long heavy plank is held so that it rests on the two beams, perpendicular to each, with its centre of gravity at B_1. The coefficients of friction between the plank and B_1 and B_2 are μ_1 and μ_2, respectively, where $\mu_1 < \mu_2$ and $\mu_1 + \mu_2 = 2\tan\alpha$.

The plank is released and slips over the beams experiencing a force of resistance from each beam equal to the limiting frictional force (i.e. the product of the appropriate coefficient of friction and the normal reaction). Show that it will come to rest with its centre of gravity over B_2 in a time

$$\pi\left(\frac{d}{g(\mu_2-\mu_1)\cos\alpha}\right)^{\frac{1}{2}} .$$

10 Three ships A, B and C move with velocities $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{u}$ respectively. The velocities of A and B relative to C are equal in magnitude and perpendicular. Write down conditions that $\mathbf{u}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ must satisfy and show that

$$\left|\mathbf{u}-\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\right|^2 = \left|\tfrac{1}{2}(\mathbf{v}_1-\mathbf{v}_2)\right|^2$$

and

$$\left(\mathbf{u}-\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\right)\cdot(\mathbf{v}_1-\mathbf{v}_2) = 0 .$$

Explain why these equations determine, for given $\mathbf{v}_1$ and $\mathbf{v}_2$, two possible velocities for C, provided $\mathbf{v}_1 \neq \mathbf{v}_2$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are equal in magnitude and perpendicular, show that if $\mathbf{u} \neq \mathbf{0}$ then $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$.

11 A uniform cylinder of radius a rotates freely about its axis, which is fixed and horizontal. The moment of inertia of the cylinder about its axis is I. A light string is wrapped around the cylinder and supports a mass m which hangs freely. A particle of mass M is fixed to the surface of the cylinder. The system is held at rest with the particle vertically below the axis of the cylinder, and then released. Find, in terms of I, a, M, m, g and θ, the angular velocity of the cylinder when it has rotated through angle θ.

Show that the cylinder will rotate without coming to a halt if $m/M > \sin\alpha$, where α satisifes $\alpha = \tan\frac{1}{2}\alpha$ and $0 < \alpha < \pi$.

Section C: Statistics

12 A bag contains b black balls and w white balls. Balls are drawn at random from the bag and when a white ball is drawn it is put aside.

(i) If the black balls drawn are also put aside, find an expression for the expected number of black balls that have been drawn when the last white ball is removed.

(ii) If instead the black balls drawn are put back into the bag, prove that the expected number of times a black ball has been drawn when the first white ball is removed is b/w. Hence write down, in the form of a sum, an expression for the expected number of times a black ball has been drawn when the last white ball is removed.

13 In a game for two players, a fair coin is tossed repeatedly. Each player is assigned a sequence of heads and tails and the player whose sequence appears first wins. Four players, A, B, C and D take turns to play the game. Each time they play, A is assigned the sequence TTH (i.e. Tail then Tail then Head), B is assigned THH, C is assigned HHT and D is assigned HTT.

(i) A and B play the game. Let p_{HH}, p_{HT}, p_{TH} and p_{TT} be the probabilities of A winning the game given that the first two tosses of the coin show HH, HT, TH and TT, respectively. Explain why $p_{\mathrm{TT}} = 1$, and why $p_{\mathrm{HT}} = \frac{1}{2}\,p_{\mathrm{TH}} + \frac{1}{2}\,p_{\mathrm{TT}}$. Show that $p_{\mathrm{HH}} = p_{\mathrm{HT}} = \frac{2}{3}$ and that $p_{\mathrm{TH}} = \frac{1}{3}$. Deduce that the probability that A wins the game is $\frac{2}{3}$.

(ii) B and C play the game. Find the probability that B wins.

(iii) Show that if C plays D, then C is more likely to win than D, but that if D plays A, then D is more likely to win than A.

14 A random variable X is distributed uniformly on $[\,0\,,\,a\,]$. Show that the variance of X is $\frac{1}{12}a^2$.

A sample, X_1 and X_2, of two independent values of the random variable is drawn, and the variance V of the sample is determined. Show that $V = \frac{1}{4}\,(X_1 - X_2)^2$, and hence prove that $2V$ is an unbiased estimator of the variance of X.

Find an exact expression for the probability that the value of V is less than $\frac{1}{12}a^2$ and estimate the value of this probability correct to one significant figure.